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A study of representations of the algebra of functions on the quantum group $GL_q(n)$

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Abstract. A q-analogue of the root system is constructed for this algebra which is similar to the root system of classical Lie algebras. It is then used to construct, in detail, a class of representations of the algebra of functions on the quantum group $GL_q(n)$, and a q-boson realization of the generators of $GL_q(n)$ is given. I also construct infinite-dimensional Hilbert-space representations of this algebra. The main result of this paper is stated in proposition 8.

1. Introduction

The problem of representations of quantum function algebras has already been studied by mathematicians [1–3]. In the works of Vaksman and Soibleman [1, 2] it was shown that the structure of these quantum function algebras resemble those of solvable Lie algebras and further it was shown that there is a correspondence between irreducible representations of quantum function algebras and symplectic leaves of the Poisson Lie groups which have been quantized to these quantum groups. (In this paper we are not concerned with co-representations of these algebras, which is a completely different problem. See [24] for the case of $SU_q(2)$.) Physicists have also considered this problem on a more explicit level and in a language more accessible to the physics community [4, 5].

My aim in this paper is to study the above problem from another point of view, namely by the introduction of a root system for this algebra which enables one to study its representations in complete analogy with those of classical Lie algebras. Using this root system I present a detailed study of a certain class of finite-dimensional representations of the quantum function algebra $GL_q(n)$. This paper is a generalization of my previous works concerning the quantum groups $GL_{q,p}(2)$ [6] and $GL_q(3)$ [7].

I remind the reader of a very well known finite-dimensional representation [8] of the generators T_{ij} of $GL_q(n)$ (see (3) below). This is the so-called R-matrix representation:

$$(T_{ij})_{\alpha,\beta} = R_{i\alpha,j\beta} \tag{1}$$

where R is the numerical R-matrix corresponding to the quantum group. There is also another R-matrix representation where R is replaced by $\hat{R} = PRP$. Here P is the permutation operator and if R is a lower triangular matrix, then $\hat{R}$ will be an upper triangular matrix. For definiteness in the following we consider the case where R is lower triangular. Such representations, however, have the obvious drawback that some of the generators are identical to the zero matrix in the represention. This is due to the triangularity of the R-matrix, and the higher the dimension of the group, the higher also the number of generators which are set identically to zero. This then means that we are not representing the totality of the algebra, but only a reduction of it, in which a large number of commutation relations have been trivialized. Therefore these representations do not reveal the true 'amount of non-commutativity' of the quantum function algebra. As an example, consider the quantum matrix $T = \binom{a\ b}{c\ d} \in GL_q(2)$. The R-matrix representation (1) sets the generator b to be identical to zero. This immediately reduces the relations to the following simple form:

$$ac = qca \qquad cd = qdc \qquad ad = da\,.$$

Particularly with regard to the last relation this stands far from the original relations of $GL_q(2)$.

The R-matrix representation for

$$T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \in GL_q(3)$$

sets the generators b, c and f equal to zero, which trivializes a lot of commutation relations. In fact, in this reduction there is only one relation which is not multiplicative, i.e. the relation between d and h, while in the original quantum matrix there are nine relations which are not multiplicative. Our aim in this paper is to study those representations in which all the commutation relations are non-trivial. For these kinds of representations which we call *complete* representations, we will show that finite-dimensional irreducible representations exist only when q is a root of unity ($q^p = 1$) and the dimensions of these representations can only be one of the following values: $p^N/2^k$ where $N = n(n-1)/2$ and $k \in \{0, 1, 2, \ldots N\}$.

We will also specify the topology of the space of states (see propposition 8). The method which we use is based on the introduction of a certain subalgebra of $GL_q(n)$ denoted by Σ_n for which one can construct finite-dimensional representations in a very straightforward way. This subalgebra is, in fact, nothing but a nice root decomposition of the original algebra. It is then shown that from each irreducible Σ_n module one can construct an irreducible $GL_q(n)$ module.

A possible relevance to physics

Usually a quantum group is associated with three kinds of equations. These are

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2}$$

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{3}$$

$$R_{12}L_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R_{12}\,. \tag{4}$$

There is also a third relation between L^+ and L^- which we supress for brevity. These equations, having no dependence on the spectral parameter, have important implications in *mathematics*. They appear in, respectively,

(M-1) theory of knots and links [21];
(M-2) defining relations of quantum function or quantum matrix algebras [10];
(M-3) defining relations of quantized universal enveloping algebras [10].

The physics enters when one puts in the spectral parameter and considers the equations

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{5}$$

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{6}$$

$$R_{12}(u-v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R_{12}(u-v)\,. \tag{7}$$

In this spectral-dependent form these equations have the following important applications in *two-dimensional physics*:

(P-1) factorizable S-matrix of $(1+1)$-dimensional quantum field theory [8, 11] where the paramter u here plays the role of rapidites of the particles;

(P-2) exactly solvable statistical mechanical models on two-dimensional lattices [9, 22]. One simply assigns to each vertex or a plaquette of a lattice a Boltzmann weight which is

$$\omega(ij;\alpha,\beta|u) = (T_{ij})_{\alpha,\beta}(u)\,. \tag{8}$$

Here, the indices of $\omega(ij;\alpha,\beta|u)$ represent the labelling of the statistical variables attached to the links or sites of the vertex or the IRF model, respectively. The c-numbers $(T_{ij})_{\alpha,\beta}(u)$ are the matrix elements of the generators $T_{ij}(u)$ in a representation. This type of Boltzmann weight guarantees the integrability of the model, since it automatically leads to a one-parameter family of commuting transfer matrices for these models.

(P-3) Quantum integrable models on the lattice. Here the operators L play the role of monodromy matrices of the lattice.

Thus going from mathematics to physics is accomplished by inserting the spectral parameter or what is technically called 'Yang–Baxterization'. We know that many numerical solutions of the Yang–Baxter equation (2) can be Yang–Baxterized [12] to become solutions of (5). The possibility of Yang–Baxterizing solutions of (4) to those of (7) has been considered in [13, 14] with the result of inventing new integrable models in $(1+1)$-dimensional field theory.

Usually the vertex or IRF models are based on the following form of the assignment of the Boltzmann weights:

$$\omega(ij;\alpha,\beta|u) = R_{i\alpha,j\beta}(u) \tag{9}$$

which may be thought of as the Yang–Baxterization of only a special kind of representation of the quantum function algebra, namely the R-matrix representation mentioned in (1). Therefore if a process of Yang–Baxterization is also found for all solutions of (3) to those of (6) then one may hope to build a more general class of integrable lattice models by using the Boltzmann weights as in (8), Boltzmann weights (9) being a very special kind of class. In this case the representations of quantum function algebras, in general, and those considered in this paper acquire physical significance.

The rest of this paper deals with representation theory. We begin by introducing a canonical root system.

2. The root system of $GL_q(n)$

The quantum matrix algebra $GL_q(n)$ [10, 15–17] is a Hopf algebra generated by unity and the elements t_{ij} of an $n\times n$ matrix T, subject to the relations [10]

$$RT_1T_2 = T_2T_1R$$

where R is the solution of the Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ corresponding to $SL_q(n)$ [18],

$$R = \Sigma_{i\neq j}e_{ii}\otimes e_{jj} + \Sigma_i q e_{ii}\otimes e_{ii} + \left(q-q^{-1}\right)\Sigma_{i<j}e_{ji}\otimes e_{ij}\,.$$

The commutation relations derived from (6) can be neatly expressed in the following way.

For any four elements a, b, c and d in the respective positions specified by rows and columns (ij), (ik), (lj) and (lk), the following relations hold:

$$\begin{aligned} &ab = qba \qquad cd = qdc \qquad ac = qca \\ &bd = qdb \qquad bc = cb \qquad ad - da = \left(q-q^{-1}\right)bc\,. \end{aligned} \tag{10}$$

For any matrix $T \in GL_q(n)$, a quantum determinant $D_q(T)$ is defined with the properties:

$$[D_qT, t_{ij}] = 0 \qquad \forall t_{ij} \in T$$
$$\Delta D_q(T) = D_q(T) \otimes D_q(T) .$$

The quantum determinant of T acquires a natural meaning as the q-analogue of the volume form when the quantum group is considered as the automorphism group on the quantum vector space associated with $GL_q(n)$ [17]. It has the following explicit expression:

$$D_q(T) = \Sigma_{i=1}^n (-q)^{i-1} t_{1i} \Delta_{1i} \tag{11}$$

where Δ_{1i} is the q-minor corresponding to t_{1i} and is defined by a similar formula.

In equation (2), $D_q(T)$ has been expanded in terms of the elements in the first row of T. Another useful expansion is in terms of the last column of T:

$$D_q(T) = \Sigma_{i=1}^n (-q)^{n-i} \Delta_{in} t_{in} . \tag{12}$$

To proceed toward constructing the root system of $GL_q(n)$ let us label the elements of the matrix T as follows:

$$T = \begin{pmatrix} . & . & . & . & . & . & Y_1 & H_1 \\ . & . & . & . & . & Y_2 & H_2 & X_1 \\ . & . & . & . & Y_3 & H_3 & X_2 & . \\ . & . & . & Y_4 & H_4 & X_3 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ Y_{n-1} & H_{n-1} & X_{n-2} & . & . & . & . & . \\ H_n & X_{n-1} & . & . & . & . & . & . \end{pmatrix} .$$

Consider the elements H_i, X_i and Y_i together with the q-minors (q-determinants of the submatrices)

$$H_{ij} = \det_q \begin{pmatrix} . & . & . & H_i \\ . & . & . & . \\ . & . & . & . \\ H_j & . & . & . \end{pmatrix} \qquad X_{ij} = \det_q \begin{pmatrix} . & . & . & X_i \\ . & . & . & . \\ . & . & . & . \\ X_j & . & . & . \end{pmatrix} \qquad Y_{ij} = \det_q \begin{pmatrix} . & . & . & Y_i \\ . & . & . & . \\ . & . & . & . \\ Y_j & . & . & . \end{pmatrix} .$$

Note. For convenience we sometimes denote H_i, X_i and Y_i by H_{ii}, X_{ii} and Y_{ii}, respectively.

Consider the subalgebra $\Sigma_n \equiv \Sigma_n^0 \oplus \Sigma_n^+ \oplus \Sigma_n^-$ where the latter are generated by the elements $H_{ij} (i \leqslant j)$, $X_{ij} (i \leqslant j)$ and Y_{ij} $i \leqslant j$, respectively.

We call the elements X_i and Y_i simple roots and the elements X_{ij} $(i < j)$ and Y_{ij} $(i < j)$ non-simple roots. As will be shown below, the generators H_i will play the role of Cartan subalgebra elements and the elements X_{ij} $(i \leqslant j)$ (resp. Y_{ij} $(i \leqslant j)$) will act as raising and lowering operators. We use the word root in a special sense, by which we mean that from representations of roots, representations of all the other elements of the quantum group can be constructed. For $GL_q(n)$ there are $N = \frac{1}{2}n(n-1)$ pairs of positive and negative roots.

The reason why constructing Σ_n modules is easy is due to the very crucial fact that almost all the relations between generators of Σ_n are multiplicative or of Heisenberg–Weyl type. By multiplicative relation between two elements x and y, we mean a relation of the form $xy = q^\alpha yx$, where α is an integer.

Remark. In the rest of this paper a multiplicative relation between x and y is indicated as $xy \approx yx$.

The important properties of Σ_n are encoded in the following propositions (see appendix A for a sketch of the proof.)

Proposition 1. For all i, j, k and l:

$$[H_{ij}, H_{kl}] = 0 \tag{13}$$

$$[X_{ij}, X_{kl}] = 0 \tag{14}$$

$$[Y_{ij}, Y_{kl}] = 0\,. \tag{15}$$

Thus Σ_n^0 and $\Sigma_n^\pm$ are three commuting subalgebras of $GL_q(n)$. For the relations between the generators of Σ_n^0 and $\Sigma_n^\pm$ we have the following.

Proposition 2.

$$H_i X_{ij} = q X_{ij} H_i \qquad \forall j \geqslant i \tag{16}$$

$$H_{j+1} X_{ij} = q X_{ij} H_{j+1} \qquad \forall i \leqslant j \tag{17}$$

$$H_k X_{ij} = X_{ij} H_k \qquad k \neq i, j+1 \tag{18}$$

$$H_{ij} X_{kl} \approx X_{kl} H_{ij} \qquad \forall i, j, k, l \tag{19}$$

with $(q \longrightarrow q^{-1}, X_{ij} \longrightarrow Y_{ij})$.

Remark. The exact coefficients in relation (10) can easily be determined (appendix A). In particular, we need the relations

$$H_{ij} X_k = X_k H_{ij} \qquad i \leqslant k \leqslant j-1 \tag{20}$$

$$H_{ij} X_{ij} = q X_{ij} H_{ij} \tag{21}$$

$$H_{i+1,j+1} X_{ij} = q X_{ij} H_{i+1,j+1} \tag{22}$$

$$[H_{i,j+1}, X_{ij}] = [H_{i+1,j}, X_{ij}] = 0\,. \tag{23}$$

The relations between elements of Σ_n^+ and Σ_n^-

Proposition 3.

$$Y_{kl} X_{ij} \approx X_{ij} Y_{kl} \qquad (k, l) \neq (i, j) \tag{24}$$

$$Y_i X_i - X_i Y_i = \left(q - q^{-1}\right) H_i H_{i+1} \tag{25}$$

$$q^{-1} Y_{ij} X_{ij} - q X_{ij} Y_{ij} = \left(q^{-1} - q\right) H_{i,j+1} H_{i+1,j}\,. \tag{26}$$

In appendix A a sketch of the proofs of these propositions is presented.

Proposition 4. For $q^p = 1$ the pth power of all the elements of Σ_n are central.

Proof. For the multiplicative relations this is obvious. The only non-multiplicative relations are (25) and (26). From (16) and (17) we have

$$H_i H_{i+1} X_i = q^2 X_i H_i H_{i+1} \tag{27}$$

using this relation and (25) we find by induction

$$Y_i X_i^n = X_i^n Y_i + (q - q^{-1}) \left\{ \frac{q^{2n} - 1}{q^2 - 1} \right\} X_i^{n-1} H_i H_{i+1} \tag{28}$$

which shows that for $q^p = 1$

$$Y_i X_i^p = X_i^p Y_i\,. \tag{29}$$

A similar argument shows that $Y_i X_i^p = X_i^p Y_i$.

For the relation (26) we use the fact that $H_{i,j+1} H_{i+1,j} X_{ij} = X_{ij} H_{i,j+1} H_{i+1,j}$. By induction from (26) we obtain

$$Y_{ij} X_{ij}^n = q^{2n} X_{ij}^n Y_{ij} + \left(1 - q^{2n}\right) X_{ij}^{n-1} H_{i,j+1} H_{i+1,j} \tag{30}$$

which again shows that

$$[Y_{ij}, X_{ij}{}^{p}] = [Y_{ij}{}^{p}, X_{ij}] = 0\,. \tag{31}$$

Definition. Let V be a Σ_n module. We call this module complete if the action of all the generators of Σ_n on it is non-trivial (i.e. not identical to zero) and call the corresponding representation of $GL_q(n)$ on V a complete representation. In the rest of this paper we are only interested in this type of representations.

Proposition 5. A Σ_n module V is complete only if all the subspaces

$$K_{ij} \equiv \{|v> \in V|,\, H_{ij}|v>= 0\}$$

are zero-dimensional.

Proof. Suppose that for some i and j $\dim K_{ij} \neq 0$. We choose a basis like $\{|e_i\rangle, i = 1, \ldots N\}$ for K_{ij}. Due to the multiplicative relation of H_{ij} with all the elements of Σ_n it is clear that for any $m \in \Sigma_n$ we have

$$H_{ij} m|e_k\rangle \approx m H_{ij}|e_k\rangle = 0\,.$$

Therefore $me_k \in K_{ij}$ which means that the basis vectors e_k transform among themselves under the action of Σ_n. Since V is assumed to be irreducible we have $K_{ij} = V$ and

$$H_{ij} V = H_{ij} K_{ij} = 0$$

which shows that V is not a complete Σ_n module.

Proposition 6.
(i) Finite-dimensional irreducible complete representations of Σ_n only exist when q is a root of unity.
(ii) Any complete Σ_n module V is also an $GL_q(n)$ module and vice versa.

Proof. (i) Suppose that q is not a root of unity let $|v_0\rangle$ be a common eigenvector of the H_{ij}'s, and consider the string of states $|l\rangle = X_1^l|v_0\rangle$; here the choice of X_1 is arbitrary, and is made for definiteness. Since $H_1 X_1 = qX_1 H_1$ we find $H_1|l\rangle = q^l|l\rangle$.

Since all these eigenvalues are different, to have a finite-dimensional representation one must have $|m\rangle \equiv X_1^m|v_0\rangle = 0$ for some m, while all the states $|l\rangle$ with $l < m$ are independent.

Now consider the string of states $\{|l'\rangle = Y_1^{l'}|m\rangle\}$. From $H_1 Y_1 = q^{-1} Y_1 H_1$ one obtains that $H_1|l'\rangle = q^{-l'+m}|l'\rangle$. Again, for finite-dimensional representations, this string of states must terminate somewhere, that is, there must exist an integer m' such that

$$Y_1^{m'}|m\rangle = 0 \qquad \text{while} \quad Y_1^{m'-1}|m\rangle \neq 0\,.$$

We will then have

$$0 = X_1 Y_1^{m'}|m\rangle = \left(Y_1^{m'} X_1 + q\left(q^{-2m'} - 1\right) Y_1^{m'-1} H_1 H_2\right)|m\rangle = q\left(q^{-2m'} - 1\right) Y_1^{m'-1} \lambda_1 \lambda_2 |m\rangle$$

where λ_1 and λ_2 are the eignevalues of H_1 and H_2 on $|m\rangle$. These eignevalues are different from zero, due to proposition 5. Noting that $Y_1^{m'-1}|m\rangle \neq 0$, we obtain $q^{-2m'} = 1$ which contradicts our earlier assumption.

(ii) The proof of this part is exactly parallel to the case of $GL_q(3)$. One uses the expressions (2) (resp. (3)) for the q-determinants Y_{ij} (resp. X_{ij}) (starting from $j = i + 1$, continuing to $j = i + 2, i + 3 \ldots$) and uses the fact that in the representation of Σ_n, all the elements H_{ij} are invertible diagonal matrices. As an example, in appendix B we carry out this procedure explicitly for the quantum group $GL_q(4)$. Note that invertibility of H_{ij}'s (due to proposition 5) is crucial here, otherwise one cannot define the actions of the remaining elements of T or V.

3. Representations

To develop the full representation theory we rescale the roots as follows:

$$h_{ij} = H_{ij} \qquad x_{ij} = \mu_{ij}{}^{-1/2} X_{ij} \qquad y_{ij} = \mu_{ij}{}^{-1/2} Y_{ij} \tag{32}$$

where $\mu_{ij} = (H_{ij} H_{i+1,j+1})$.

I have verified by many examples that with this redefinition the root system is completely disentangled into mutually commuting pairs, while all the relations between H_{ij} and X_{ij} (Y_{ij}) remain intact. Instead of (24)–(26) one will have

$$[x_{ij}, y_{kl}] = 0 \qquad (k, l) \neq (i, j) \tag{33}$$

$$q^{-1} x_i y_i - q y_i x_i = (q^{-1} - q)\mathbf{1} \tag{34}$$

$$[x_{ij}, y_{ij}] = (q - q^{-1}) \frac{h_{i,j+1} h_{i+1,j}}{h_{ij} h_{i+1,j+1}} . \tag{35}$$

From these relations one can also obtain the more general relations,

$$y_i x_i{}^l = q^{-2l} x_i{}^l y_i + (1 - q^{-2l}) x_i{}^{l-1} \tag{36}$$

$$y_{ij} x_{ij}{}^l = x_{ij}{}^l y_{ij} + q(q^{-2l} - 1) x_{ij}{}^{l-1} \frac{h_{i,j+1} h_{i+1,j}}{h_{ij} h_{i+1,j+1}} . \tag{37}$$

With this redefinition the only structure constants of the algebra are the coefficients between the h_{ij} and x_{ij}. Table 1 shows these structure constants for $GL_q(4)$.

Consider a common eigenvector of h_{ij} which we denote by $|\mathbf{0}\rangle$ with eigenvalues $h_{ij}|\mathbf{0}\rangle = \lambda_{ij}|\mathbf{0}\rangle$ and construct an $(N = \frac{1}{2}n(n-1))$-dimensional hypercube of states

$$W = \left\{ |l\rangle = \prod_{i,j} (x_{ij})^{l_{ij}} |\mathbf{0}\rangle, 0 \leqslant l_{ij} \leqslant p - 1 \right\} \tag{38}$$

where l is a vector $l = \sum_{i \leqslant j} l_{ij} e_{ij}$ in the lattice. From equation (10) all the states of W are eigenstates of h_{ij}

$$h_{ij}|l\rangle = q^{c_{ij}(l)} \lambda_{ij} |l\rangle . \tag{39}$$

The parameters $c_{ij}(l)$ can easily be calculated by using the structure constants (see appendix B where the case of $GL_q(4)$ is considered as an example).

Each positive root generates one direction of this hypercube. Because of (14) we have

$$x_{ij}|l\rangle = |l + e_{ij}\rangle . \tag{40}$$

Table 1. The structure constants of $GL_q(4)$, i.e. $h_{12} x_{12} = q x_{12} h_{12}$.

	x_1	x_2	x_3	x_{12}	x_{23}	x_{13}
h_1	q	1	1	q	1	q
h_2	q	q	1	1	q	1
h_3	1	q	q	q	1	1
h_4	1	1	q	1	q	q
h_{12}	1	q	1	q	q	q
h_{23}	q	1	q	q	q	1
h_{34}	1	q	1	q	q	q
h_{13}	1	1	q	1	q	q
h_{24}	q	1	1	q	1	q

Since ${x_{ij}}^p$ is central we can set its value on W equal to a c-number which for later convenience we denote by $(\lambda_{ij}\lambda_{i+1,j+1})^{-1}\eta_{ij}$. Therefore we have

$$x_{ij}|(p-1)e_{ij}\rangle = (\lambda_{ij}\lambda_{i+1,j+1})^{-1}\eta_{ij}|\mathbf{0}\rangle \,. \tag{41}$$

The last relation says much more. We need some terminology. Denote by F_{ij}^0 and F_{ij}^1 the two faces which are perpendicular to the vector e_{ij}, passing through the origin and the point $(p-1)e_{ij}$, respectively. Now if v is any vector in F_{ij}^1 then by commutativity of all x_{ij}'s we have

$$x_{ij}|(p-1)e_{ij}+v\rangle = (\lambda_{ij}\lambda_{i+1,j+1})^{-1}\eta_{ij}|v\rangle \,. \tag{42}$$

In this way when η_{ij} is non-zero the generator x_{ij} folds each face F_{ij}^1 back onto the face F_{ij}^0. Define the action of y_{ij} on $|\mathbf{0}\rangle$ by

$$y_{ij}|\mathbf{0}\rangle = \alpha_{ij}|(p-1)e_{ij}\rangle \,. \tag{43}$$

By the same reasoning as in the case of x_{ij} one can show that when α_{ij} is non-zero the generator y_{ij} folds the face F_{ij}^0 back onto the face F_{ij}^1, i.e. for any vector u lying in F_{ij}^0

$$y_{ij}|u\rangle = \alpha_{ij}|(p-1)e_{ij}+u\rangle \,. \tag{44}$$

We now calculate the action of the negative roots on the other states of W. Thanks to the commutation relations (33) one can calculate the action of any root like y_k on any state as follows:

$$y_k|l\rangle = y_k\left(\prod_i x_i^{l_i}\right)|\mathbf{0}\rangle = \prod_{i\neq k} x_i^{l_i}\, y_k {x_k}^{l_k}|\mathbf{0}\rangle \,. \tag{45}$$

For simplicity of notation, in this equation we have represented any positive (resp. negative) root by the symbol x_k (resp. y_k) and have not distinguished between simple and non-simple roots. One then uses (36) and (37) to complete the calculation. The result is

$$y_i|l\rangle = \left(q^{-2l_i}\alpha_i\eta_i + \left(1-q^{-2l_i}\right)\right)|l-e_i\rangle \tag{46}$$

$$y_{ij}|l\rangle = \left(\alpha_{ij}\eta_{ij} + q\left(1-q^{-2l_{ij}}\right)s_{ij}\right)|l-e_{ij}\rangle \tag{47}$$

where $s_{ij} = (\lambda_{i,j+1}\lambda_{i+1,j})/(\lambda_{ij}\lambda_{i+1,j+1})$. This shows that each y_{ij} acts as a lowering operator in the direction e_{ij} of the hypercube.

It remains to determine the parameters λ_{ij}. Clearly calculation of these parameters by direct expansion of h_{ij} is cumbersome. Instead we proceed as follows. Denote by $A(i_1, i_2\ldots|j_1, j_2\ldots)$ the q-determinant obtained from a quantum matrix A by deleting the rows $i_1, i_2\ldots$ and columns $j_1, j_2, \ldots$ [3]. Then we conjecture that the following identity is true:

$$A(n|n)A(1|1) - qA(n|1)A(1|n) = A(1,n|1,n)\,\mathrm{Det}_q A \,. \tag{48}$$

The validity of this equation can be verified by direct computation for low-dimensional $GL_q(n)$ matrices. It may also be possible to derive it by combining the relations obtained in [3]. We will give further justification for it using the conjugation properties of Σ_n. Equation (48) implies the following relation in Σ_n:

$$Y_{ij}X_{ij} = qH_{ij}H_{i+1,j+1} + H_{i,j+1}H_{i+1,j} \,. \tag{49}$$

For q on the unit circle the elements of T allow the following conjugation:

$$t_{ij}^\dagger = t_{ij} \,. \tag{50}$$

This results in the following conjugation properties in Σ_n:

$$X_{ij}^\dagger = X_{ij} \qquad Y_{ij}^\dagger = Y_{ij} \qquad H_{ij}^\dagger = H_{ij} \,. \tag{51}$$

One can then conjugate both sides of (49) to obtain

$$X_{ij}Y_{ij} = q^{-1}H_{ij}H_{i+1,j+1} + H_{i,j+1}H_{i+1,j} \,. \tag{52}$$

Combination of (49) and (52) then leads to (26), which we know to be true (see appendix A).

In terms of the rescaled generators, relation (44) takes the following form:

$$x_i y_i = 1 + q\frac{h_{i,i+1}}{h_i h_{i+1}} \tag{53}$$

$$x_{ij} y_{ij} = 1 + q\frac{h_{i,j+1}h_{i+1,j}}{h_{i,j}h_{i+1,j+1}} \,. \tag{54}$$

Now these relations help us to determine the parameters λ_{ij}. Acting on the state $|\mathbf{0}\rangle$ by both sides of (53) and (54) we obtain

$$\alpha_i\eta_i = \lambda_i\lambda_{i+1} + q\lambda_{i,i+1} \tag{55}$$

$$\alpha_{ij}\eta_{ij} = \lambda_{ij}\lambda_{i+1,j+1} + q\lambda_{i,j+1}\lambda_{i+1,j} \tag{56}$$

or

$$\lambda_{i,i+1} = q^{-1}(\alpha_i\eta_i - \lambda_i\lambda_{i+1}) \tag{57}$$

$$\lambda_{i,j+1} = q^{-1}\frac{(\alpha_{ij}\eta_{ij} - \lambda_{i,j}\lambda_{i+1,j+1})}{\lambda_{i+1,j}} \,. \tag{58}$$

Let us call λ_{ij} the weights of the representation and call each $\lambda_{i,i+k}$ a weight at level k. Equations (57) and (58) express the weights at each level in terms of the weights at the lower level (see the appendix for the example of $GL_q(4)$).

4. Types of representations

We complete our analysis of representations of $GL_q(n)$ by a discussion on the various types of representations. Each representation is defined by the n^2 parameters α_{ij}, η_{ij} and λ_i. The type of representation depends on the values of the parameters α_{ij} and η_{ij}. More precisely we have the following.

Proposition 8. The dimensions of the irreducible complete representations of $GL_q(n)$ can only be one of the following values: $p^N/2^k$ where $N = n(n-1)/2$ and $k \in \{0, 1, 2, \ldots N\}$. For each k the topology of the space of states is $(S^1)^{\times(N-k)} \times [0, 1]^{\times(k)}$ (i.e. an N-dimensional torus for $k = 0$ and an N-dimensional cube for $k = N$).

Proof. Our style of proof is a generalization of the one given in [6, 7] for the case of $GL_{q,p}(2)$ and $GL_q(3)$, respectively.

Let V be a $GL_q(n)$ module with dimension d. Depending on the values of the parameters α_{ij} and η_{ij} three cases can happen.

Case (a). $\alpha_{ij} \neq 0 \neq \eta_{ij}$, $\forall i, j$.

In this case d cannot be greater than p^N, otherwise the cube W will span an invariant submodule which contradicts the irreducibility of V. The dimension of V cannot be less than p^N either, since this means that the length of one of the sides of the cube W (say in the ith direction) must be less than p. Therefore there must exist a positive integer $r < p$ such that $x_i{}^r|0\rangle = 0$, which means that $\eta_i|0\rangle = x_i{}^{p-r}x_i{}^r|0\rangle = 0$, contradicting the original assumption. The topology of the space of states in this case is an N-dimensional torus $\left(S^{1\times N}\right)$.

Case (b). For some (ij) $\alpha_{ij} \neq 0$, but $\eta_{ij} = 0$ or vice versa.

In this case the representation is semicyclic in the ijth direction.

Case (c). For some (ij) $\alpha_{ij} = \eta_{ij} = 0$.

In this case the representation has a highest and a lowest weight in the ijth direction.

If $d < P^N$ there must exist an integer like $r < p$ such that $x_{ij}^r|0\rangle = 0$ and $x_{ij}^l|0\rangle \neq 0$ for $l < r$. Now denote $x_{ij}^r|0\rangle$ by u_0 and consider the string of states $y_{ij}^{l'}u_0$. This string of states must terminate somewhere. That is, there must exists an integer like r' such that $y_{ij}^{r'}u_0 = 0$ and $y_{ij}^{r'-1}u_0 \neq 0$. Therefore

$$0 = x_{ij}y_{ij}^{r'}u_0 = \left(y_{ij}^{r'}x_{ij} + q(q^{-2r'} - 1)y_{ij}^{r'-1}\frac{h_{i,j+1}h_{i+1,j}}{h_{ij}h_{i+1,j+1}}\right)u_0 = q(q^{-2r'} - 1)s_{ij}u_0$$

which means that $q^{2r'} = 1$ or $r' = \frac{1}{2}p$. r' is in fact the length of the edge of the cube W in the ijth direction, the other dedges being of length p. The dimension of V is in this case $\frac{1}{2}p^N$. The topology of the space of states is in this case $[0, 1] \times S^{1^{\otimes N-1}}$. By repeating this analysis for other pairs of the parameters the assertion is proved.

The classical ($q = 1$) commutative case

In the classical limit ($q = 1$) one has $p = 1$. Therefore the hypercube W will be one-dimensional; $\{W = \text{span }\{|0\rangle\}\}$.

The action of all the elements of T on this state will be representated by pure numbers, as expected, since in this limit we are talking about irreducible representations of a commutative algebra.

5. The Hilbert space representations

In this section we restrict ourselves to the case when q is real and consider the infinite-dimensional highest-weight representations.

For real q the algebra $GL_q(n)$ admits the following conjugation:

$$t_{ij}^* = t_{n+j-1,n+i-1} \,. \tag{59}$$

Note. This should not be confused with the conjugation $t_{ij}{}^* = t_{ij}$ introduced in (59) which is only valid for q on the unit circle.

Equation (59) has a simple meaning. It says that the conjugate of each element of T is its own mirror image with respect to the opposite diagonal. One need not do any lengthy calculation to prove that the conjugation (59) is consistent with the commutation relations of the algebra. One simply draws an arbitrary rectangle with elements a, b, c and d on its corners satisfying (10) and reflect it with respect to the opposite diagonal and check that the new relations between a^*, b^*, c^* and d^* are again of type (10). With the involution (59), $GL_q(n)$ is turned into a $*$ algebra but not into a $*$ Hopf algebra, since the relation $(\Delta a)^* = \Delta(a^*)$ does not hold. This then means that the category of representations is not closed under a tensor product. However, a very close relation exists, that is $(\Delta a)^* = \Delta'(a^*)$ where $\Delta' \equiv \sigma \circ \Delta$ is the opposite co-multiplication. To see this, note that

$$\begin{aligned}(\Delta t_{ij})^* &= (t_{ik} \otimes t_{kj})^* = (t_{ik}^* \otimes t_{kj}^*) = (t_{n+1-k,n+1-i} \otimes t_{n+1-j,n+1-k}) \\ &= \sigma(t_{n+1-j,n+1-k} \otimes t_{n+1-k,n+1-i}) = \Delta'(t_{n+1-j,n+1-i}) = \Delta'(t_{ij}^*)\end{aligned}$$

where a sum over k from 1 to n is implied in all the above formulae. It may therefore be possible to still use this type of conjugation in an effective way, since the tensor product representations constructed by Δ and Δ' are equivalent and can be intertwined by the R-matrix.

It is not difficult to see that under conjugation the following relations hold:

$$X_{ij}^* = Y_{ij} \qquad Y_{ij}^* = X_{ij} \qquad H_{ij}^* = H_{ij}\,. \tag{60}$$

Using this star structure it is now possible to represent $GL_q(n)$ on a Hilbert space.

Assume that the vacuum vector is normalized:

$$\langle 0|0\rangle = 1\,.$$

We compute the norm of the state $|l\rangle$. From (38) we have

$$|l\rangle = \prod_{i,j} x_{ij}{}^{l_{ij}}|0\rangle\,. \tag{61}$$

Note that x_i is a simple root and $x_{ij}(i < j)$ is a non-simple root. Their commutation relations with their corresponding negative roots are given by (34) and (35), respectively. We then have

$$\langle l|l\rangle = \langle 0|\prod_{i,j} y_{ij}{}^{l_{ij}} \prod_{i,j} x_{ij}{}^{l_{ij}}|0\rangle\,. \tag{62}$$

Thanks to the simple commutation relations (33) we have

$$\begin{aligned}\langle l|l\rangle &= \langle 0|\prod_{i,j} y_{ij}{}^{l_{ij}} x_{ij}{}^{l_{ij}}|0\rangle \\ &= \langle 0|\prod_{i<j} y_{ij}{}^{l_{ij}} x_{ij}{}^{l_{ij}} \prod_{i} y_i{}^{l_i} x_i{}^{l_i}|0\rangle\,.\end{aligned} \tag{63}$$

We now use the relation (36) to obtain

$$y_i{}^m x_i{}^m|0\rangle = \left(1-q^{-2m}\right) y_i{}^{m-1} x_i{}^{m-1}|0\rangle = \cdots = \prod_{k=1}^{m}\left(1-q^{-2k}\right)|0\rangle \tag{64}$$

where we have used the fact that y_i anihilates the vacuum.

We also use (37) to obtain

$$y_{ij}{}^m x_{ij}{}^m|0\rangle = q\left(q^{-2m}-1\right) y_{ij}{}^{m-1} x_{ij}{}^{m-1} \frac{h_{i,j+1}h_{i+1,j}}{h_{ij}h_{i+1,j+1}}|0\rangle\,. \tag{65}$$

The right-hand side can be considerably simplified by noting that if we act on the vacuum vector by both sides of (52) we obtain

$$\frac{h_{i,j+1}h_{i+1,j}}{h_{ij}h_{i+1,j+1}}|0\rangle = -q^{-1}|0\rangle\,. \tag{66}$$

In fact this means that in the infinite-dimensional representations the parameters s_{ij} are all equal to $-q^{-1}$. Therefore we will have

$$y_{ij}{}^m x_{ij}{}^m|0\rangle = \left(1-q^{-2m}\right) y_{ij}{}^{m-1} x_{ij}{}^{m-1}|0\rangle = \cdots = \prod_{k=1}^{m}\left(1-q^{-2k}\right)|0\rangle\,. \tag{67}$$

Denoting $\prod_{k=1}^{m}\left(1-q^{-2k}\right)$ by $(m)!$ we obtain

$$\langle l|l\rangle = \prod_{i,j}(l_{ij})!\,. \tag{68}$$

In order to have a positive norm for the states we restrict ourselves to $q^2 \geqslant 1$. In the classical limit ($q = 1$) where the algebra $GL_q(n)$ becomes a commutative algebra one expects that the irreducible representations will be one-dimensional. In the above case this is reflected in the fact that in this limit all the states except the vacuum have zero norm and are decoupled from the Hilbert space. Using the above type of analysis it is also straightforward to show that all the different states are orthogonal to each other.

Normalizing the state $|l\rangle$ to $|l) = (1/\prod_{i,j}(l_{ij})!)|l\rangle$ we will have

$$x_{ij}|l) = \sqrt{1-q^{2(l_{ij}+1)}}|l+e_{ij}) \tag{69}$$

$$y_{ij}|l) = \sqrt{1-q^{2(l_{ij})}}|l-e_{ij}) \tag{70}$$

$$h_{ij}|l) = q^{c_{ij}(l)}\lambda_{ij}|l) \tag{71}$$

where the parameters $c_{ij}(l)$ were defined previously (see (39)).

6. Q-boson realization

One can construct an infinite-dimensional representation (q-analogue of the Verma module) by setting all $\alpha_i = 0$ and relaxing all the conditions of preiodicity (q real). It is then very easy to determine the q-boson realization of all the generators of Σ_n and hence of $GL_q(n)$.

The q-boson algebra [19–21] B_q is generated by three elements $a, a^\dagger$ and N satisfying the relations

$$aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N} \tag{72}$$

$$q^{\pm N}a = q^{\mp 1}aq^{\pm N} \qquad\qquad q^{\pm N}a^\dagger = q^{\pm 1}a^\dagger q^{\pm N}\,. \tag{73}$$

A more useful form of the algebra is obtained if one replaces the above equations by the following pair of relations:

$$aa^\dagger = [N+1] \qquad a^\dagger a = [N] \tag{74}$$

where the symbol $[N]$ as usual stands for $(q^N - q^{-N})/(q-q^{-1})$, with N being a number or an operator.

On the q Fock space F_q spanned by the states $|n\rangle \equiv a^{\dagger^n}|0\rangle$ the action of the generators are

$$a^\dagger|n\rangle = |n+1\rangle \tag{75}$$

$$a|n\rangle = [n]_q|n-1\rangle \tag{76}$$

$$N|n\rangle = n|n\rangle\,. \tag{77}$$

Consider N commuting q-bosons (i.e. $a_i, a^\dagger{}_i, N_i; i = 1\ldots N$) and their representation on the q Fock space $F_q^{\otimes N}$. Then if Ψ is the natural isomorphism from W to $F_q^{\otimes N}$, satisfying

$$\Psi : |l\rangle \longrightarrow \prod_{i=1}^{N} a_i{}^{l_i}|0\rangle \tag{78}$$

the induced representation Ψ is defined by [13]

$$\Psi^*(g) = \Psi \circ g \circ \Psi^{-1} \qquad \forall g \in \mathrm{End}\ W\,. \tag{79}$$

We will then have the following n^2 parameter family of q-boson realization of the quantum group $GL_q(n)$:

$$x_i = a_i^\dagger \qquad\qquad x_{ij} = a_{ij}^\dagger \tag{80}$$

$$y_i = (q-q^{-1})a_iq^{-N_i} \qquad y_{ij} = (q-q^{-1})a_{ij}q^{-N_{ij}} \tag{81}$$

$$h_i = \lambda_i q^{C_i(N)} \qquad\qquad h_{ij} = q^{C_{ij}(N)}\lambda_{ij}\,. \tag{82}$$

7. Discussion

As remarked in the introduction, the R-matrix representations of $GL_q(n)$ have the shortcoming that they actually represent a rather strong reduction of $GL_q(n)$, obtained by imposing the additional relations $t_{ij} = 0 \; \forall \, j > i$. The representations considered in this paper seem to be at the opposite extreme. That is, by definition it seems that they do not allow any reduction. The price we have paid for representing the whole algebra is the high dimension of the representations. Certainly a large class of representations lies between these two extremes. A natural question is whether it is possible to at least partially relax the conditions of our definition and obtain representations of reductions of the algebra by starting from the representations presented in this paper? Due to the simple properties of the basis and the complete similarity that it has with the Cartan–Weyl basis of classical Lie algebras, I think it is possible to obtain a lot of other representations by this method, and if one does this, systematically and carefully, at some stage the condition of q being a root of unity will be relaxed and perhaps one may also obtain the R-matrix representations in this way.

For example, we can consider the quotients of $GL_q(n)$ in which any number of the following q-determinants vanishes:

$$H_1, H_{12}, H_{13}, \dots H_{1n} \qquad H_n, H_{n-1,n}, H_{n-2,n}, \dots H_{1n} \,.$$

It is essential to note that setting any of these q-determinants to zero:

(i) Is consistent with the commutation relations. The reader may verify, by looking at some low-dimensional quantum matrices, that other reductions of this type with other indices for H are not consistent with commutation relations. For example, in $GL_q(3)$ one cannot only set the relation $H_2 = 0$. Note that setting a q-determinant equal to zero does not imply that its indivdual elements have been nullified. The latter reduction requires more relations.

(ii) Still allows us to extract a representation of $GL_q(n)$ from that of Σ_n, since we do not need invertibility of these elememts in this extraction process. (See part 2 of the proof of proposition 6 and appendix B.)

(iii) Furthermore, setting the corresponding vacuum eigenvalues of these operators, i.e. $\lambda_1, \lambda_{12}, \lambda_{13}, \dots \lambda_{1n}$ and $\lambda_{n-1,n}\lambda_{n-2,n}\lambda_{n-3,n}, \dots \lambda_{1n}$ equal to zero does not make the expression of other λ_{ij}'s in (57) and (58) singular, since exactly these eigenvalues do not appear in the denominator of the right-hand sides of these equations.

Acknowledgments

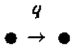
I would like to thank all my colleagues in the physics department of IPM for very valuable discussions. I also thank A Morozov for interesting comments made during his visit to IPM.

Appendix A. Commutation relations in Σ (proofs of propositions 1–3)

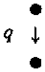
Reference [3] presents some of the commutation relations between the q-determinants of the submatrices of $T \in GL_q(n)$ (more precisely those submatrices which are obtained by deleting one row and column from the original matrix). However, most of the relations that we need are not among the relations studied in [3]. Therefore in the following we present a graphical method in contrast to the analytical method of [3], to obtain those relations that we need. Our method and results are not a substitute for those of [3], both are their complement.

In what follows, any element of the matrix T will be shown by a $\bullet$ and any q-minor of any size by a square. The positions of the dots or squares represent their r positions in the matrix T, and the order of the elements in a multiplicative relation is shown by an arrow, and the factor which is obtained when one reverses the sence of the arrow is indicated on the arrow.

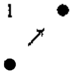
Thus the multiplicative relations in (10) are depicted as follows:



(A1*a*)



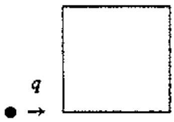
(A1*b*)



(A1*c*)

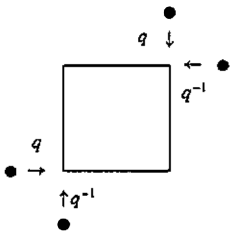
The first basic fact is presented in the following lemma.

Lemma 9.

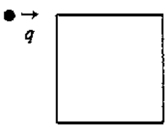


Proof. Expand the determinant and note that all of the relations are of type (A1*c*) except the ones on the lower edge, which are of type (A1*a*).

We combine diagram 1 with three similar relations in the following diagram.



Lemma 10.



Proof. Let the minor be $n \times n$. For $n = 2$, direct calculation verifies the statement. We use induction on n. Consider figure A1. Writing Δ_{n+1} as

$$\Delta_{n+1} = \Sigma d_i C_i$$

where C_i is the co-factor of d_i in Δ_{n+1} and passing a through d_i we have

$$a\Delta_{n+1} = \Sigma\big(d_i a + \big(q - q^{-1}\big)bc_i\big)C_i\,.$$

We now use the assumption of induction ($aC_i = qC_i a$) and the property of the determinant ($\Sigma a_i C_i = 0$) to arrive at the final result.

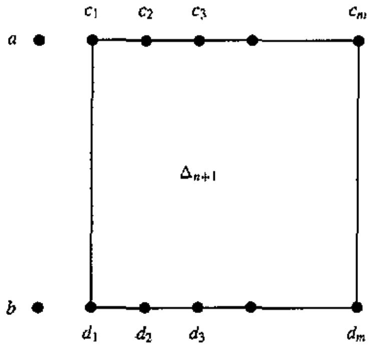
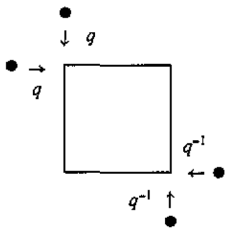
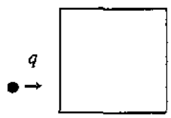


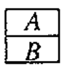
Figure A1. Proof of lemma 10.

It is combined with three other relations in the following diagram.



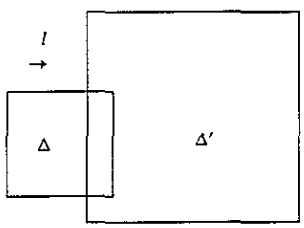
Lemma 11.



Proof. Write the q-minor as 

where A is the q-minor lying just above the •, which symbolically means that the q-minor is the sum of the products of the elements of A and q-co-factors in B. Passing the • through A gives the factor 1 and passing it through B gives q.

Corollary.



$$\Delta\Delta' = q^l \Delta'\Delta .$$

Proof. Expand the left-hand minor and use lemma 11.

The • (resp. the small minor) can be in other similar positions as in the previous two lemmas, with appropriate factors of q or q^{-1} (resp. q^l or q^{-l})

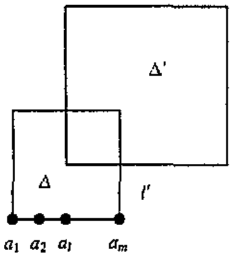
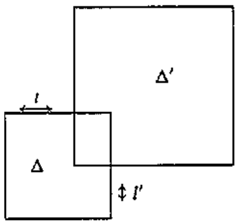


Figure A2. Proof of lemma 12.

Lemma 12.



$$\Delta\Delta' = q^{l-l'}\Delta'\Delta .$$

Proof. Consider figure A2. We use induction on l'. For $l' = 0$ the result is true. Assume that it is true for l' and write Δ as $\Delta = \Sigma_{i=1}^{m} a_i C_i$, where C_i is the q-co-factor of a_i in Δ, and use the results of the previous lemmas, i.e.

$$\begin{array}{lll} a_i\Delta' = \Delta' a_i & C_i\Delta' = q^{-l'}q^{l-1}\Delta' c_i & 1 \leqslant i < l \\ a_i\Delta' = q^{-1}\Delta' a_i & C_i\Delta' = q^{-l'}q^{l}\Delta' c_i & l \leqslant i \leqslant m \end{array}$$

from which we obtain the result for $l' + 1$.

Important remark. In the matrix T there are many more positions of q-minors which give rise to very complicated commutation relations. But in Σ there is none (as the reader can verify) other than those between X_{ij} and Y_{ij}, which we now compute exactly.

Proof of the last relation in proposition 3. Consider figure A3. Write $H_{i,j+1}$ as $H_{i,j+1} = a_1C_1 + a_2C_2 + \cdots$ where $C_1 = X_{ij}$ is the q-co-factor of a_1 in the big matrix. $H_{i,j+1}$, being the determinant of the big matrix, commutes with a_1. On the other hand

$$a_1 H_{i,j+1} = a_1(a_1 X_{ij} + \Sigma_{i\geqslant 2} a_i C_i) .$$

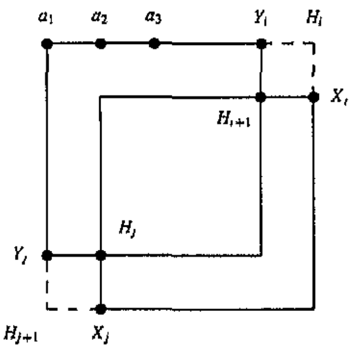


Figure A3. Proof of proposition 3.

We now use the fact that for $i \geqslant 2, a_1 a_i = q a_i a_1, a_1 C_i = q C_i a_1$ (lemma 10) and pass a_1 through $\Sigma a_i C_i$ to find

$$a_1 H_{i,j+1} = a_1^2 X_{ij} + q^2 (H_{i,j+1} - a_1 X_{ij}) a_1 .$$

Multiplying both sides from the left by a^{-1} we obtain

$$a_1 X_{ij} - q^2 X_{ij} a_1 = \left(1 - q^2\right) H_{i,j+1} . \tag{A2}$$

(Note: direct calculations which do not need invertibility of a_1 confirm this equation.) Now we expand Y_{ij} in

$$Y_{ij} X_{ij} = \left(a_1 \hat{C}_1 + \Sigma_{k \geqslant 2} a_k \hat{C}_k\right) X_{ij}$$

where $\hat{C}_k$ is the co-factor of a_k in the matrix Y_{ij}. Note that $\hat{C}_1 = H_{i+1,j}$.

From lemmas 10 and 11 we have for $k \geqslant 2$,

$$\left(a_k \hat{C}_k\right) X_{ij} = q^2 X_{ij} \left(a_k \hat{C}_k\right) .$$

Therefore

$$Y_{ij} X_{ij} = a_1 X_{ij} H_{i+1,j} + q^2 X_{ij} (Y_{ij} - a_1 H_{i+1,j}) .$$

Combining this with (83) we obtain the final result, i.e.

$$q^{-1} Y_{ij} X_{ij} - q X_{ij} Y_{ij} = \left(q^{-1} - q\right) H_{i,j+1} H_{i+1,j} .$$

In this section we have derived the general commutation relations between those q-minors of T which generate Σ. By looking at particluar posititions of these minors one can verify propositions 1–3.

Appendix B. An example: the case of $GL_q(4)$

The structure constants of $GL_q(4)$ are indicated in table 1. Conseqently we obtain the following actions:

$$h_1|l\rangle = q^{l_1+l_{12}+l_{13}} \lambda_1 |l\rangle \qquad h_2|l\rangle = q^{l_1+l_2+l_{13}} \lambda_2 |l\rangle$$
$$h_3|l\rangle = q^{l_2+l_3+l_{12}} \lambda_3 |l\rangle \qquad h_4|l\rangle = q^{l_3+l_{13}+l_{23}} \lambda_4 |l\rangle$$
$$h_{12}|l\rangle = q^{l_2+l_{12}+l_{23}+l_{13}} \lambda_{12} |l\rangle \qquad h_{23}|l\rangle = q^{l_1+l_3+l_{12}+l_{23}} \lambda_{23} |l\rangle$$
$$h_{34}|l\rangle = q^{l_2+l_{12}+l_{23}+l_{13}} \lambda_{34} |l\rangle \qquad h_{13}|l\rangle = q^{l_3+l_{23}+l_{13}} \lambda_{13} |l\rangle$$
$$h_{24}|l\rangle = q^{l_1+l_{12}+l_{13}} \lambda_{24} |l\rangle .$$

The weights λ_{ij} are determined from (49) and (50) to be

$$\lambda_{12} = q^{-1}(\alpha_1 \eta_1 - \lambda_1 \lambda_2) \qquad \lambda_{23} = q^{-1}(\alpha_2 \eta_2 - \lambda_2 \lambda_3)$$
$$\lambda_{34} = q^{-1}(\alpha_3 \eta_3 - \lambda_3 \lambda_4) \qquad \lambda_{13} = q^{-1} \frac{(\alpha_{12} \eta_{12} - \lambda_{12} \lambda_{23})}{\lambda_2}$$
$$\lambda_{24} = q^{-1} \frac{(\alpha_{23} \eta_{23} - \lambda_{23} \lambda_{34})}{\lambda_3} \qquad \lambda_{14} = q^{-1} \frac{(\alpha_{13} \eta_{13} - \lambda_{13} \lambda_{24})}{\lambda_{23}} .$$

In the following we carry out explicitly the process of reconstruction of $GL_q(4)$ from Σ_4.

Let us label the elements of $T \in GL_q(4)$ as follows:

$$T = \begin{pmatrix} p & l_1 & Y_1 & H_1 \\ l_2 & Y_2 & H_2 & X_1 \\ Y_3 & H_3 & X_2 & m_1 \\ H_4 & X_3 & m_2 & n \end{pmatrix} .$$

Here we have

$$X_{12} = H_2 m_1 - q X_1 X_2 \qquad X_{23} = H_3 m_2 - q X_2 X_3$$
$$Y_{12} = l_1 H_2 - q Y_1 Y_2 \qquad Y_{23} = l_2 H_3 - q Y_2 Y_3$$

from which we obtain

$$m_1 = H_2{}^{-1}(X_{12} + q X_1 X_2) \qquad m_2 = H_3{}^{-2}(X_{23} + q X_2 X_3)$$
$$l_1 = (Y_{12} + q Y_1 Y_2) H_2{}^{-1} \qquad l_2 = (Y_{23} + q Y_2 Y_3) H_3{}^{-1}\,.$$

We also have

$$X_{13} = H_{23} n - q(Y_2 m_2 - q H_2 X_3) m_1 + q^2 X_{23} X_1$$
$$Y_{13} = p H_{23} - q l_1 (l_2 X_2 - q H_2 Y_3) + q^2 Y_1 Y_{23}$$

from which we obtain

$$n = H_{23}^{-1}\left\{X_{13} + q(Y_2 m_2 - q H_2 X_3) m_1 + q^2 X_{23} X_1\right\}$$
$$p = \left\{Y_{13} + q l_1 (l_2 X_2 - q H_2 Y_3) + q^2 Y_1 Y_{23}\right\} H_{23}^{-1}\,.$$

These equations show that once the action of Σ_4 is known on V the action of $GL_q(4)$ can be determined uniquely.

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